On Generalizations of Fourier Transform

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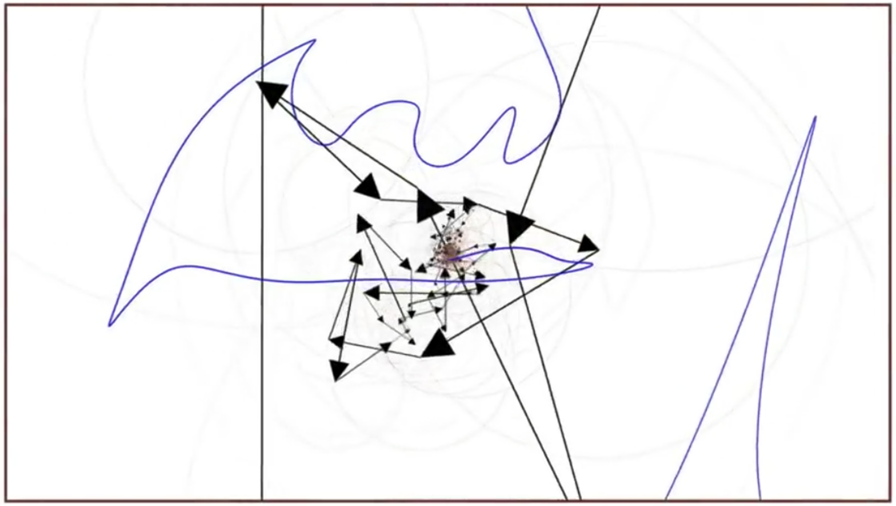
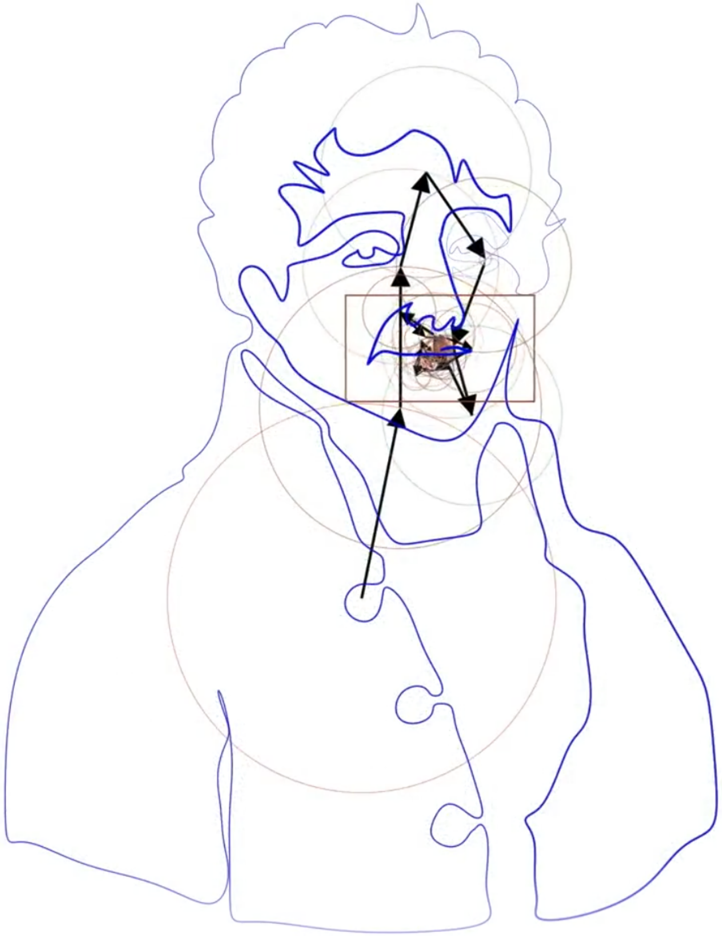




Fourier with a signal processing mindset

Useful way to represent functions via complex exponentials $e^{j\omega t}$ - the Fourier Inversion Theorem,

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \langle f, \frac{1}{\sqrt{2\pi}} e^{j\omega t} \rangle,$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{j\omega t} d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \langle f, \frac{1}{\sqrt{2\pi}} e^{j\omega t} \rangle e^{j\omega t} d\omega.$$

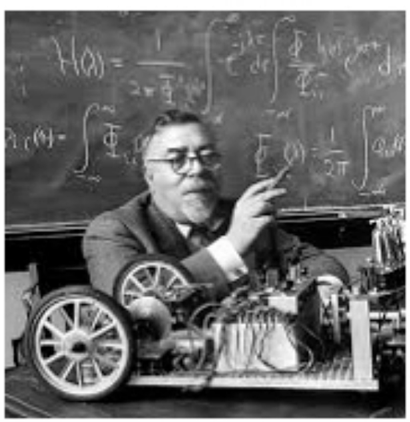
What's the reason to go beyond Fourier?

- Complex exponentials — the building blocks of the Fourier transform — are the eigenfunctions of linear time-invariant systems, $e^{j\omega t} \to H(j\omega)e^{j\omega t}$.
- In practice, many systems and physical phenomena are modeled as linear and time-varying / non-stationary.
  $\to$ radar, sonar, holography, quantum optics, non-destructive testing
- Polynomial phase models are used for modeling time-varying systems.
  $\to$ basis functions of the form $e^{j\varphi(t)}$
  $\to$ e.g., quadratic chirps specified by $\varphi(t) = a_2 t^2 + a_1 t + a_0$
- Study of polynomial phase representations led to the development of unitary transformations that generalize the Fourier transform.
  $\to$ the Fractional Fourier Transform (FrFT) [Wiener'29/Condon'37]
  $\to$ the Linear Canonical Transform (LCT) [Moshinsky/Quesne'71]
  $\to$ the Special Affine Fourier Transform (SAFT) [Abe/Sheridan'94]

Fractionalization of the Fourier transform

Edward Condon

Norbert Wiener



IMMERSION OF THE FOURIER TRANSFORM IN A CONTINUOUS GROUP OF FUNCTIONAL TRANSFORMATIONS

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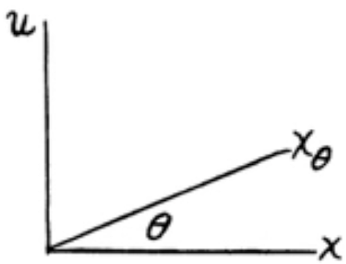
Communicated January 20, 1937

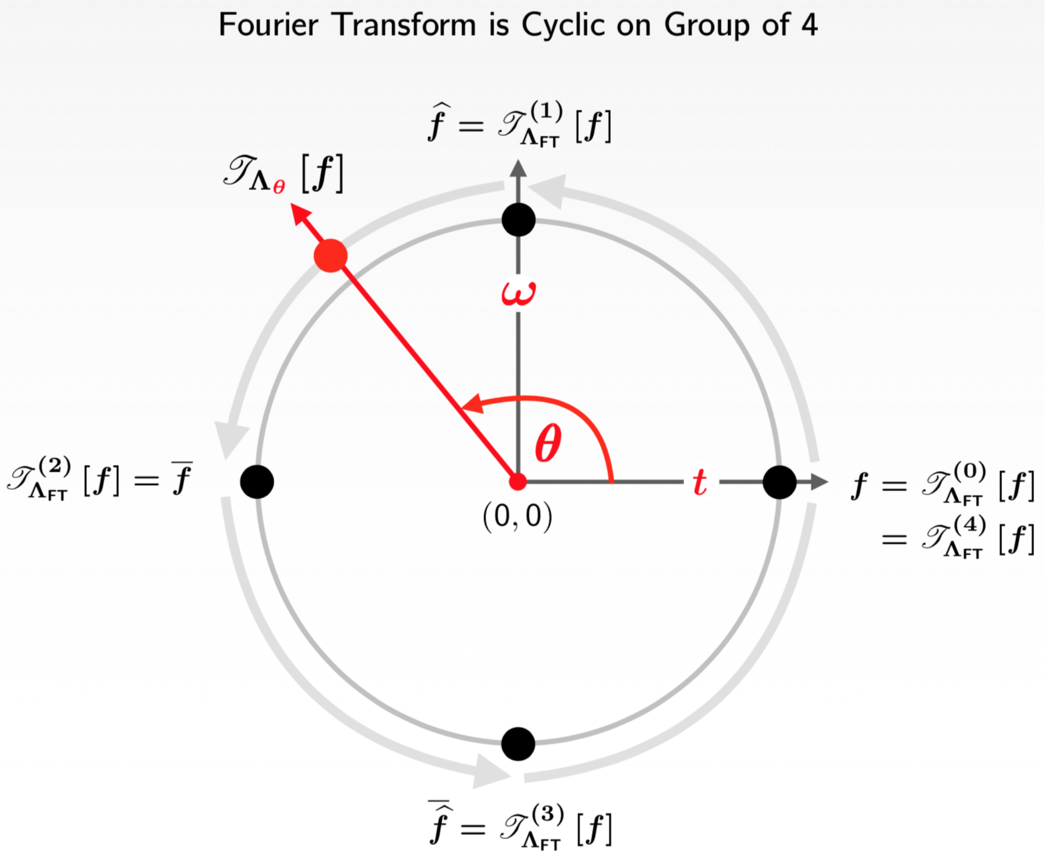
Hence the operation F generates a cyclic group of order 4 which is isomorphic with the group of rotations of a plane about a fixed point through multiples of a right angle. Now every continuous group of transformations is generated by an Hermitian operator, and conversely every Hermitian operator generates a group of unitary transformations.† Hence there exists a continuous group of functional transformations containing the ordinary Fourier transforms as a subgroup. In this paper the continuous group is explicitly found. It will not, however, be necessary to make further reference in the work that follows to the general immersion theory.

It is convenient to introduce a group space as shown in the figure in which the notation x_θ is assigned to the argument of a function which is generated out of $f(x)$ by application to it of the functional transformation F_θ. In this notation $F_\theta f(x)$ will be a transformed function of the argument x_θ which we may write as $f_\theta(x_\theta)$. Evidently x is x_0 and $f(x)$ is more explicitly $f_0(x_0)$ in this notation.

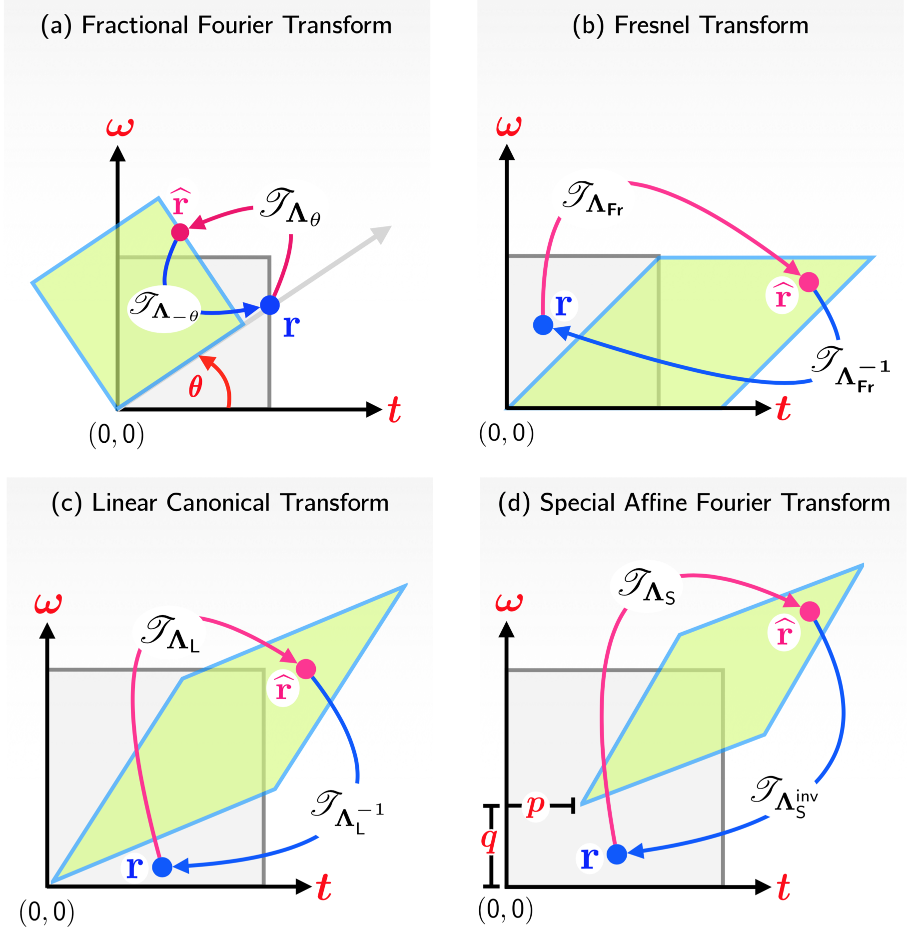


Fractionalization of the Fourier transform

Consider the Fourier operator composition: $\mathscr{F}^{(0)}_{\mathbf{\Lambda}_{\mathrm{FT}}} = f, \quad \mathscr{F}^{(n)}_{\mathbf{\Lambda}_{\mathrm{FT}}} = \left(\mathscr{F}^{(n-1)}_{\mathbf{\Lambda}_{\mathrm{FT}}} \circ \mathscr{F}^{(n)}_{\mathbf{\Lambda}_{\mathrm{FT}}}\right),$ $n \in \mathbb{Z}^{+}$



Generalization of the Fourier transform



Generalization of the Fourier transform

The Special Affine Fourier Transform (SAFT) of a signal is a mapping $\mathscr{F}_{\mathrm{SAFT}} : f \to f_{\boldsymbol{\Lambda}_{\mathrm{S}}}$ which is defined by an integral transformation parametrized by a matrix $\boldsymbol{\Lambda}_{\mathrm{S}}$,

$$\mathscr{F}_{\boldsymbol{\Lambda}_{\mathrm{S}}}[f] = \widehat{f}_{\boldsymbol{\Lambda}_{\mathrm{S}}}(\omega) = \begin{cases} \langle f, \kappa_{\boldsymbol{\Lambda}_{\mathrm{S}}}(\cdot, \omega)\rangle, & b \neq 0, \\ \sqrt{d} e^{j\frac{cd}{2}(\omega - p)^2 + j\omega q} f(d(\omega - p)), & b = 0. \end{cases}$$

- $\boldsymbol{\Lambda}_{\mathrm{S}}^{(2\times 3)}$ is the SAFT parameter matrix,

  $$\boldsymbol{\Lambda}_{\mathrm{S}} = \left[\begin{array}{cc|c} a & b & p \\ c & d & q \end{array}\right] \equiv \left[\begin{array}{c|c} \boldsymbol{\Lambda}_{\mathrm{L}} & \boldsymbol{\lambda} \end{array}\right],$$

  with an offset vector $\boldsymbol{\lambda} = [p \quad q]^T$ representing displacement p and modulation q, and $|\boldsymbol{\Lambda}_{\mathrm{L}}| = 1$.

- $\kappa_{\boldsymbol{\Lambda}_{\mathrm{S}}}$ is the SAFT kernel based on a complex exponential of quadratic form. Let $\boldsymbol{r} = [t \quad \omega]^T$ denote the time-frequency coordinates, then

  $$\kappa_{\boldsymbol{\Lambda}_{\mathrm{S}}}(\boldsymbol{r}) = K^*_{\boldsymbol{\Lambda}_{\mathrm{S}}} \exp(-j(\boldsymbol{r}^T \boldsymbol{U} \boldsymbol{r} + \boldsymbol{v}^T \boldsymbol{r})), \quad \text{where}$$

  $$\boldsymbol{U} = \frac{1}{2b}\begin{bmatrix} a & -1 \\ -1 & d \end{bmatrix}, \quad \boldsymbol{v} = \frac{1}{b}\begin{bmatrix} p \\ bq - dp \end{bmatrix}, \quad K_{\boldsymbol{\Lambda}_{\mathrm{S}}} = \frac{1}{\sqrt{j2\pi b}} \exp\left(j\frac{dp^2}{2b}\right).$$

Generalization of the Fourier transform

<table>
<tr><th>SAFT Parameters ($\boldsymbol{\Lambda}_{\mathsf{S}}$)</th><th>Corresponding Transform</th></tr>
<tr><td>$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \end{array}\right] = \boldsymbol{\Lambda}_{\mathsf{FT}}$</td><td>Fourier Transform (FT)</td></tr>
<tr><td>$\left[\begin{array}{cc|c} 0 & 1 & p \\ -1 & 0 & q \end{array}\right] = \boldsymbol{\Lambda}^{\mathsf{O}}_{\mathsf{FT}}$</td><td>Offset Fourier Transform</td></tr>
<tr><td>$\left[\begin{array}{cc|c} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{array}\right] = \boldsymbol{\Lambda}_{\theta}$</td><td>Fractional Fourier Transform (FrFT)</td></tr>
<tr><td>$\left[\begin{array}{cc|c} \cos\theta & \sin\theta & p \\ -\sin\theta & \cos\theta & q \end{array}\right] = \boldsymbol{\Lambda}^{\mathsf{O}}_{\theta}$</td><td>Offset Fractional Fourier Transform</td></tr>
<tr><td>$\left[\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \end{array}\right] = \boldsymbol{\Lambda}_{\mathsf{L}}$</td><td>Linear Canonical Transform (LCT)</td></tr>
<tr><td>$\left[\begin{array}{cc|c} 1 & b & 0 \\ 0 & 1 & 0 \end{array}\right] = \boldsymbol{\Lambda}_{\mathsf{Fr}}$</td><td>Fresnel Transform</td></tr>
<tr><td>$\left[\begin{array}{cc|c} 0 & \jmath & 0 \\ \jmath & 0 & 0 \end{array}\right] = \boldsymbol{\Lambda}_{\mathsf{LT}}$</td><td>Laplace Transform (LT)</td></tr>
<tr><td>$\left[\begin{array}{cc|c} \jmath\cos\theta & \jmath\sin\theta & 0 \\ \jmath\sin\theta & -\jmath\cos\theta & 0 \end{array}\right]$</td><td>Fractional Laplace Transform</td></tr>
<tr><td>$\left[\begin{array}{cc|c} 1 & \jmath b & 0 \\ \jmath & 1 & 0 \end{array}\right]$</td><td>Bilateral Laplace Transform</td></tr>
<tr><td>$\left[\begin{array}{cc|c} 1 & -\jmath b & 0 \\ 0 & 1 & 0 \end{array}\right], b \geq 0$</td><td>Gauss–Weierstrass Transform</td></tr>
<tr><td>$\frac{1}{\sqrt{2}}\left[\begin{array}{cc|c} 0 & e^{-\jmath\pi/2} & 0 \\ -e^{-\jmath\pi/2} & 1 & 0 \end{array}\right]$</td><td>Bargmann Transform</td></tr>
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<table>
<tr><th>SAFT Parameters ($\boldsymbol{\Lambda}_{\mathsf{S}}$)</th><th>Corresponding Signal Operation</th></tr>
<tr><td>$\left[\begin{array}{cc|c} 1/\alpha & 0 & 0 \\ 0 & \alpha & 0 \end{array}\right] = \boldsymbol{\Lambda}_{\alpha}$</td><td>Time Scaling</td></tr>
<tr><td>$\left[\begin{array}{cc|c} 1 & 0 & \tau \\ 0 & 1 & 0 \end{array}\right] = \boldsymbol{\Lambda}_{\tau}$</td><td>Time Shift</td></tr>
<tr><td>$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & \xi \end{array}\right] = \boldsymbol{\Lambda}_{\xi}$</td><td>Frequency Shift/Modulation</td></tr>
</table>

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<tr><th>SAFT Parameters ($\boldsymbol{\Lambda}_{\mathsf{S}}$)</th><th>Corresponding Optical Operation</th></tr>
<tr><td>$\left[\begin{array}{cc|c} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{array}\right] = \boldsymbol{\Lambda}_{\theta}$</td><td>Rotation</td></tr>
<tr><td>$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ \tau & 1 & 0 \end{array}\right] = \boldsymbol{\Lambda}_{\tau}$</td><td>Lens Transformation</td></tr>
<tr><td>$\left[\begin{array}{cc|c} 1 & \eta & 0 \\ 0 & 1 & 0 \end{array}\right] = \boldsymbol{\Lambda}_{\eta}$</td><td>Free Space Propagation</td></tr>
<tr><td>$\left[\begin{array}{cc|c} e^{\beta} & 0 & 0 \\ 0 & e^{-\beta} & 0 \end{array}\right] = \boldsymbol{\Lambda}_{\beta}$</td><td>Magnification</td></tr>
<tr><td>$\left[\begin{array}{cc|c} \cosh\alpha & \sinh\alpha & 0 \\ \sinh\alpha & \cosh\alpha & 0 \end{array}\right] = \boldsymbol{\Lambda}_{\eta}$</td><td>Hyperbolic Transformation</td></tr>
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Example

Let the parameter matrix be

$$\boldsymbol{\Lambda}_\theta = \left[\begin{array}{cc|c} a & b & p \\ c & d & q \end{array}\right] = \left[\begin{array}{cc|c} cos(\theta) & sin(\theta) & 0 \\ -sin(\theta) & cos(\theta) & 0 \end{array}\right].$$

Then we have

$$\begin{aligned} \boldsymbol{r}^T\boldsymbol{U}\boldsymbol{r} + \boldsymbol{v}^T\boldsymbol{r} &= [t \quad \omega]\frac{1}{2sin(\theta)}\begin{bmatrix} cos(\theta) & -1 \\ -1 & cos(\theta) \end{bmatrix}\begin{bmatrix} t \\ \omega \end{bmatrix} + \frac{1}{sin(\theta)}[0 \quad 0]\begin{bmatrix} t \\ \omega \end{bmatrix} \\ &= \frac{1}{2}(t^2+\omega^2)cot(\theta) - j\omega t csc(\theta) \end{aligned}$$

and the corresponding transform is the Fractional Fourier Transform (FrFT)

$$\widehat{f}_\theta(\omega) = K_{\boldsymbol{\Lambda}_\theta}\int_{-\infty}^{\infty} f(t)\exp\left(\frac{j}{2}(t^2+\omega^2)cot(\theta) - j\omega t csc(\theta)\right)dt.$$

The Inverse SAFT

SAFT matrix maps time-frequency coordinates $\boldsymbol{r} = [t \quad \omega]^T$ into its affine transformed version,

$$\begin{bmatrix} t \\ \omega \end{bmatrix} \xrightarrow{\text{SAFT}} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} t \\ \omega \end{bmatrix} + \begin{bmatrix} p \\ q \end{bmatrix} \equiv \boldsymbol{r} \xrightarrow{\text{SAFT}} \boldsymbol{\Lambda}_{\mathrm{L}} \boldsymbol{r} + \boldsymbol{\lambda}.$$

Hence, the inverse SAFT is defined by affine transform that allows for the mapping

$$\begin{bmatrix} at + b\omega + p \\ ct + d\omega + q \end{bmatrix} \xrightarrow{\text{Inverse SAFT}} \begin{bmatrix} t \\ \omega \end{bmatrix}.$$

It can be shown that the inverse parameter matrix $\boldsymbol{\Lambda}_{\mathrm{S}}^{\mathrm{inv}}$ is

$$\boldsymbol{\Lambda}_{\mathrm{S}}^{\mathrm{inv}} := \left[\begin{array}{cc|c} +d & -b & bq - dp \\ -c & +a & cp - aq \end{array}\right] \equiv \left[\begin{array}{c|c} \boldsymbol{\Lambda}_{\mathrm{L}}^{-1} & -\boldsymbol{\Lambda}_{\mathrm{L}}^{-1} \boldsymbol{\lambda} \end{array}\right].$$

Thus, the inverse transform (iSAFT) is defined as an SAFT with matrix $\boldsymbol{\Lambda}_{\mathrm{S}}^{\mathrm{inv}}$,

$$\mathscr{F}_{\boldsymbol{\Lambda}_{\mathrm{S}}^{\mathrm{inv}}}[\widehat{f}] = f(t) = K_{\boldsymbol{\Lambda}_{\mathrm{S}}^{\mathrm{inv}}} \langle \widehat{f}_{\boldsymbol{\Lambda}_{\mathrm{S}}}, \kappa_{\boldsymbol{\Lambda}_{\mathrm{S}}^{\mathrm{inv}}}(\cdot, t) \rangle,$$

where $\kappa_{\boldsymbol{\Lambda}_{\mathrm{S}}^{\mathrm{inv}}}(\omega, t) = \kappa^*_{\boldsymbol{\Lambda}_{\mathrm{S}}}(t, \omega)$ and $K_{\boldsymbol{\Lambda}_{\mathrm{S}}^{\mathrm{inv}}} = \exp\left(\frac{j}{2}(cdp^2 + abq^2 - 2adpq)\right)$.

Subspace of SAFT-bandlimited functions

- By SAFT-bandlimited function, we refer to a signal whose SAFT has bounded support.

Definition [Bhandari/Zayed]

Let $\Delta = b/\Omega$. The family of functions

$$V_{\mathrm{BL}}^{\boldsymbol{\Lambda}_{\mathrm{S}}} = \mathrm{span}\left\{ \varphi_n(t) = \frac{1}{\Delta} e^{-j\frac{a(t^2-(n\Delta)^2)}{2b}} e^{-jp\frac{t-n\Delta}{b}} \mathrm{sinc}\left(\frac{t}{\Delta} - n\right) \right\}_{n\in\mathbb{Z}}$$

forms an orthonormal subspace of SAFT-bandlimited functions with maximum admissible frequency $\omega_{max} = \pi\Omega = b\pi/\Delta$.

Extension of Shannon's sampling theorem

- Any function that belongs to the subspace of SAFT-bandlimited functions may be exactly recovered via its orthogonal projection onto this subspace, i.e., $f = \mathscr{P}_{\varphi} f$.

Theorem (Shannon's Sampling in SAFT domain) [Stern'07/Xiang et al.'12]

$$\mathscr{P}_{\varphi} f = \sum_{n\in\mathbb{Z}} \langle f, \varphi_n \rangle \varphi_n(t) = \frac{e^{-j\frac{at^2}{2b}}}{\Delta} \sum_{n\in\mathbb{Z}} f(n\Delta) e^{j\frac{a(n\Delta)^2}{2b}} e^{-jp\frac{t-n\Delta}{b}} \operatorname{sinc}\left(\frac{t}{\Delta} - n\right).$$

Intuition: You can recover the original function exactly from its samples!

One-bit sampling of bandlimited functions

- In conventional systems, Shannon-Nyquist approach is used. Sampling amounts to

$$\mathscr{S}_{\mathrm{Sh}} : f(t) \to f[n] = f(nT), \quad n \in \mathbb{Z}, \quad T > 0.$$

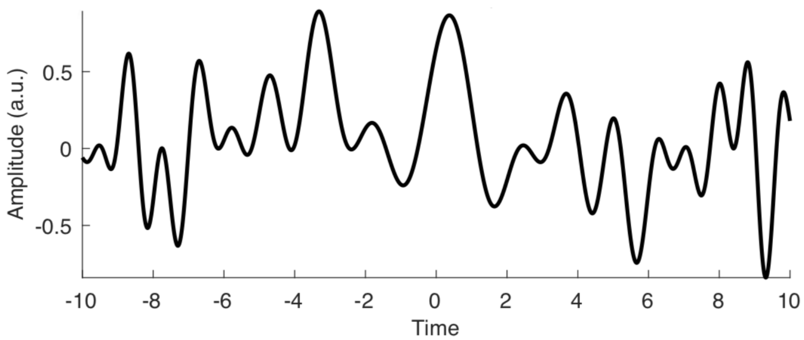
- Our approach is based on

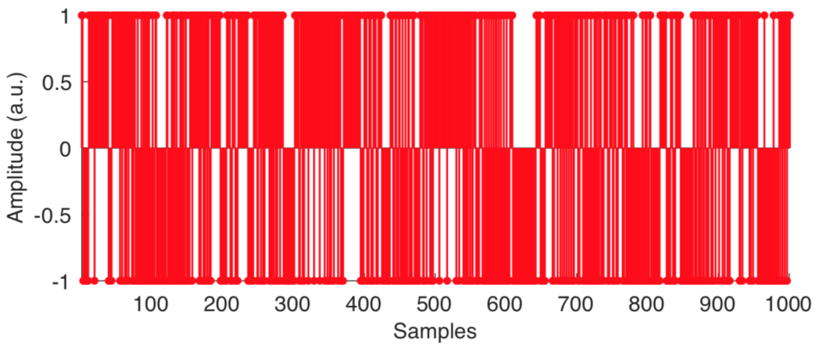
$$\mathscr{S}_{\mathrm{1B}} : f(t) \to q[n] \in \{-1, 1\}$$

which is implemented by

$$q[n] = \mathrm{sgn}\left(u[n-1] + f[n]\right)$$
$$u[n] = u[n-1] + f[n] - q[n].$$

One-bit sampling of bandlimited functions





$$\underbrace{\begin{aligned} q[n] &= \mathrm{sgn}(u[n-1] + f[n]) \\ u[n] &= u[n-1] + f[n] - q[n] \end{aligned}}_{One-Bit\ \ Sampling} \xrightarrow{\text{Recovery}} \underbrace{\widetilde{f}(t) = \frac{1}{\lambda} \sum_{n \in \mathbb{Z}} q[n] \varphi\left(\frac{t}{T} - \frac{n}{\lambda}\right)}_{Interpolation}$$

Remarkably, the recovery follows essentially Shannon's sampling formula, where λ denotes the oversampling factor and $\varphi(t)$ is a $\lambda\Omega$-bandlimited function.

One-bit sampling of bandlimited functions

$$\underbrace{\Delta[n] = \delta[n] - \delta[n-1]}_{\textit{Finite Difference Filter}}$$

$$L > 1, \quad v^{[L]}[n] = (v * \cdots * v)[n]$$

$$\widehat{V}^{[L]}(\omega) = (1 - e^{j\omega})^L = 2(sin(\omega/2))^L$$

Time domain

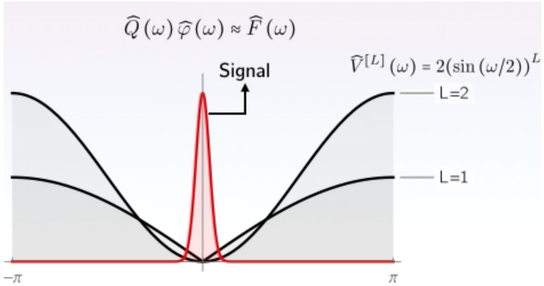
$$\underbrace{q[n]}_{One-bit} = \underbrace{f[n]}_{Data} - \underbrace{(u * v)[n]}_{Noise}$$

Fourier domain

$$\underbrace{\widehat{Q}(\omega)}_{One-bit} = \underbrace{\widehat{F}(\omega)}_{Data} - \underbrace{\widehat{U}(\omega)\widehat{V}(\omega)}_{Noise}$$

$$\underbrace{|f(t) - \widetilde{f}(t)|}_{Max-Error\ \ Bound} \leqslant \frac{1}{\lambda^L}\|\varphi^{(L)}\|_{L_1}\max_n|u[n]|, \quad L, \lambda > 1$$

[Daubechies/DeVore'03]



Larger $\lambda \Rightarrow$ Signal is concentrated around origin. Smaller the error.

One-bit sampling in FrFT domain

- Assume we have a function f that is Ω–bandlimited in the FrFT domain or,

$$f \in \mathcal{B}_{\Omega}^{\theta} \Leftrightarrow \widehat{f_{\theta}}(\omega) = \widehat{f_{\theta}}(\omega)\,\mathbf{1}_{|\omega| \leqslant \Omega}(\omega).$$

- f is no longer bandlimited in Fourier domain.
- Convolution of two functions <u>does not</u> amount to a multiplication of their spectrums in the FrFT domain: $\mathscr{F}_{\mathbf{\Lambda}_{\mathrm{S}}}[f * g] \neq \mathscr{F}_{\mathbf{\Lambda}_{\mathrm{S}}}[f]\mathscr{F}_{\mathbf{\Lambda}_{\mathrm{S}}}[g]$.

$\Rightarrow$ conventional noise shaping fails!

One-bit sampling in FrFT domain
[Bhandari/Graf/Krahmer/Zayed'20]

- Generalized One-Bit Sampling

  For any $f \in \mathcal{B}_{\Omega}^{\theta} \Leftrightarrow \widehat{f_\theta}(\omega) = \widehat{f_\theta}(\omega)\mathbf{1}_{|\omega|\leqslant\Omega}(\omega)$,

$$
\begin{aligned}
u[0] &\in (-1,1), \\
u[n] &= f\left(\tfrac{nT}{\lambda}\right) + e^{-j\frac{2n-1}{2}\left(\frac{T}{\lambda}\right)^2\cot\theta} u[n-1] - q_\theta[n], \\
q_\theta[n] &= \operatorname{csgn}\left(e^{-j\frac{2n-1}{2}\left(\frac{T}{\lambda}\right)^2\cot\theta} u[n-1] + f\left(\tfrac{nT}{\lambda}\right)\right),
\end{aligned}
$$

  where

$$
\operatorname{csgn}(z) = \operatorname{sgn}(\operatorname{Re}(z)) + j\operatorname{sgn}(\operatorname{Im}(z)).
$$

- Recovery from One-Bit Samples

  Given $q_\theta[n]$, $f \in \mathcal{B}_{\Omega}^{\theta}$ is approximated by

$$
\widetilde{f}(t) = \frac{e^{-j\frac{t^2}{2}\cot\theta}}{\lambda} \sum_{n\in\mathbb{Z}} q_\theta[n]\, e^{j\frac{(nT)^2}{2\lambda^2}\cot\theta} \varphi\left(\frac{t}{T} - \frac{n}{\lambda}\right).
$$

A basic bound on reconstruction error

- Our strategy entails the same error bound as in Fourier domain.

Proposition (Error bound)
Let $f \in \mathcal{B}_{\Omega}^{\theta}$ be such that $\|f\|_{L_\infty} \leqslant 1$ and $\lambda > 1$. Given one-bit samples, the reconstruction is bounded by the following inequality,

$$\|f - \widetilde{f}\|_{L_\infty} \leqslant \frac{1}{\lambda}\|\varphi^{(1)}\|_{L_1}.$$

- This bound is based on the accumulation of noise in the low-pass FrFT spectrum,

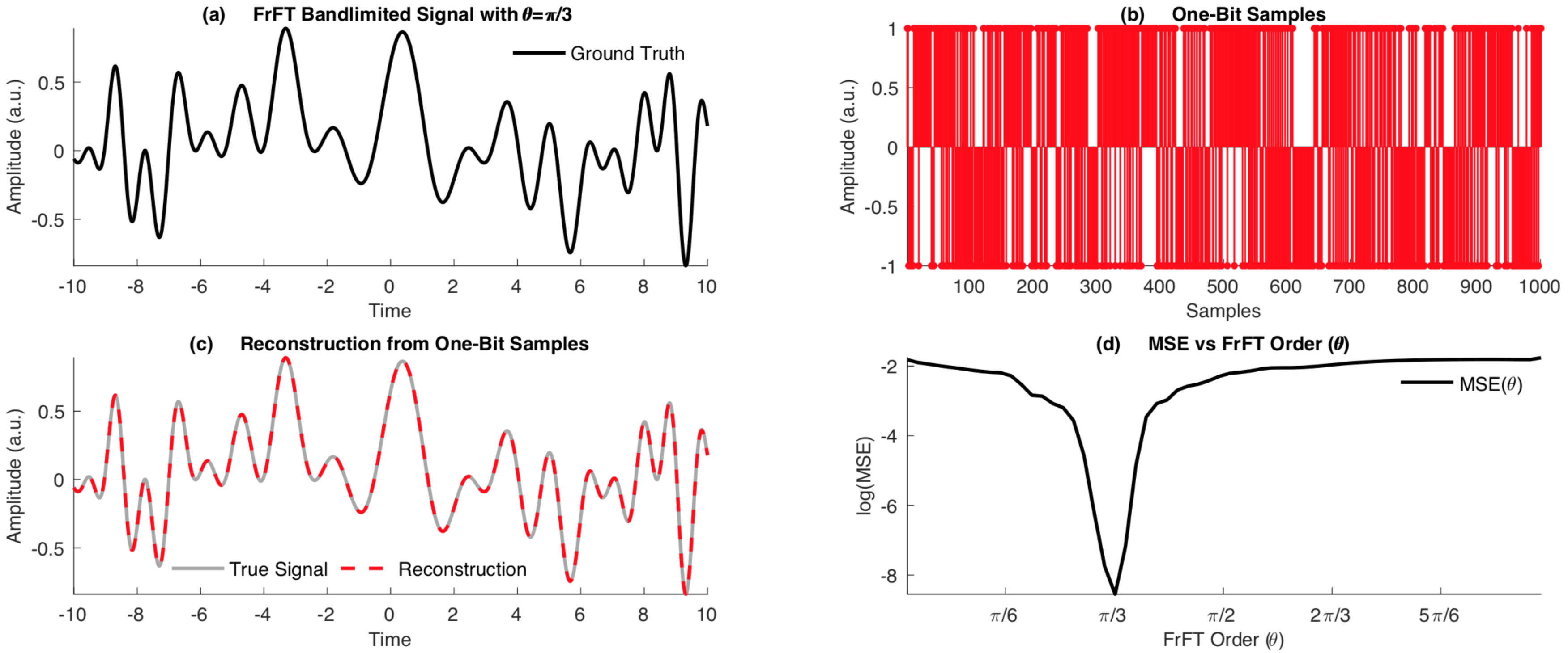
$$\underbrace{f(t) - \widetilde{f}(t)}_{\text{Error}} = \left(\frac{e^{-j\frac{t^2}{2}\cot\theta}}{\lambda}\right) \underbrace{\sum_{n\in\mathbb{Z}} \left(\vec{u} * \Delta\right)[n]\, \varphi\left(\frac{t}{T} - \frac{n}{\lambda}\right)}_{\text{Low-pass contamination due to noise}}.$$

Experimental demonstration

- Consider a signal generated by chirp-modulating a mixture of sinusoids,

$$f(t) = \left(\tfrac{2}{5}\cos(t) + \tfrac{1}{2}\sin\left(\tfrac{5}{2}t + \tfrac{\pi}{6}\right)\right) e^{-j\frac{t^2}{2}\cot\frac{\pi}{3}}.$$

- Note that $f \in \mathcal{B}_{\Omega}^{\pi/3}$ and clearly $f \notin \mathcal{B}_{\Omega}^{\pi/2}$ (Fourier).



Conclusions

- The FrFT and the SAFT generalize the Fourier transform.
  $\longrightarrow$ Due to this flexibility, they have found a number of theoretical and practical applications.
- We have introduced a new signal representation for the FrFT domain.
  $\longrightarrow$ We presented a one-bit sampling and reconstruction method in the FrFT domain.
  $\rightarrow$ The reconstruction error bound is the same as Fourier domain bound.

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Thank you!